**Homework 10**



**P20.2.8** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.8. Evaluate *f*(*t*)\**g*(*t*) for all *t*. Verify the result using multiplication of polynomials.

**Solution:** After folding one of the functions around the vertical axis, and shifting by t, the values at the breakpoints are obtained for the various time intervals/



For 0 ≤ *t* ≤ 1:

At *t* = 1, *y* = 9.



For 1 ≤ *t* ≤ 2:

At *t* = 2, *y* = 3×2 – 3×2 = 0.

For 2 ≤ *t* ≤ 3:



At *t* = 3, *y* = 1×3 – 2×2 + 3×1 = 2.



For 0 ≤ *t* – 3 ≤ 1, or 3 ≤ *t* ≤ 4:

At *t* = 4, *y* = -1×2 + 2×1 = 0.



For 1 ≤ *t* – 3 ≤ 2, or 4 ≤ *t* ≤ 5:

*A*t *t* = 5, *y* = 1× = 1.

For 2 ≤ *t* – 3 ≤ 3, or 5 ≤ *t* ≤ 6:

*A*t *t* = 6, *y* = 0.



The breakpoints are as shown. To use Matlab, enter f = [3,2,1] and

g = [3, -2, 1] corresponding to the levels of *f*(*t*) and *g*(*t*). Then enter: conv(f,g). Matlab returns: 9, 0, 2, 0, 1, corresponding to the breakpoints.

**P20.2.10** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.10, where *g*(*t*) = sin*π*(t - 1), 1 ≤ *t* ≤ 2, and *g*(*t*) = 0 elsewhere. Determine *f*(*t*)\**g*(*t*) for all *t*. Verify by convolving with step functions.



**Solution:** sin*π*(*t* – 1) = -sin*πt*. When *g*(*λ*) is folded with respect to the vertical axis and shifted to the right by *t*, *y*(*t*) = 0 for *t* ≤ 1.

For 1 ≤ *t* ≤ 2, the graphical construction will be as shown, and **.



For 2 ≤ *t* ≤ 3, *y*(*t*) remains constant and equals **.

For 3 ≤ *t* ≤ 4, ** **.



*y*(*t*) = 0 for *t* ≥ 4 and will be as shown.

Alternatively, *y*(*t*) can be derived as the convolution of sinusoidal and step functions. Thus, *f*(*t*) = *u*(*t*) – *u*(*t* – 2), and *g*(*t*) = ** + *.* It follows that: *y*(*t*) = ** + *.* These terms will be convolved using Equation 20.4.2. Thus, with *b* = 1 and *a* = 0,



**= **

**. It follows that *y*(*t*) = 0 for 0 ≤ *t* ≤ 1 and **, as before, for 1 ≤ *t* ≤ 2.

For 2 ≤ *t* ≤ 3, * * **. Adding this to the previous result gives **, as before.

For 3 ≤ *t* ≤ 4, -**= ** **. Adding this to the previous result gives **, as before.

For *t* ≥ 4, -** = ** **. Adding this to the previous result gives *y*(*t*) = 0, as before.



**P20.2.12** Given *f*(*t*) in Figure P20.2.12. Determine the convolution integral when *f*(*t*) is convolved with itself. Verify the result analytically and by convolution of step functions.

**Solution:** *Graphical*: When one pulse is reflected about the vertical axis, it remains in position. It has to be shifted to the left by *t* = -*τ*/2 to start the integration. As the pulse is shifted to the right, the area of integration increases linearly with *t*, reaching a maximum at *t* =0, when the area under the product is *τ*/2. For larger *t*, the area decreases linearly, becoming 0 at *t* = +*τ*/2. The convolution integral is thus a triangular pulse of amplitude *τ*/2 and width *τ*, centered at the origin. The analytical expression of the convolution integral is:  =



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*Analytical*: **, where the limits of integration are determined by the range of λ over which the integrand is not zero. Thus, *,* where *t* > -*τ*/2; **, *t* > -*τ*/2; or, **.

*,* where *t* > *τ*/2; * , t* > -*τ*/2; or, **.

*,* where *t* > 0; **= -*t*, *,*; or, **.

*,* where *t* > 0; **= -*t*, *,*; or, **. Adding *y*1(*t*), *y*2(*t*), *y*3(*t*) and *y*4(*t*) gives the same *y*(*t*) as before.

*Convolution of step functions*: The step functions will be convolved using Equation 20.4.2. Thus, **;

**(*t* + τ/2), *t* >

-*τ*/2; or, **, as before.

** (*t* – τ/2), *t* >

*τ*/2; or, **, as before.

*-t*, *t* > 0,

r, **, as before.

* t*, *t* > 0,

or, **, as before.

**P20.2.13** Given *f*(*t*) and *g*(*t*) in Figure P20.2.13. (a) Evaluate *f*(*t*)\**g*(*t*); (b) If the impulse response of a circuit is , determine the response of the circuit to *f*(*t*).



**Solution:** (a) When *f*(*t*) is folded around the vertical axis, it coincides with *g*(*t*). It must be shifted in the negative direction by 3 units to start the integration at *t* = -3 units. As *f*(-*λ*) is shifted to the right, the convolution integral increases linearly, reaching a maximum at *t* = 0, when the two functions coincide, then decreasing linearly to zero after 3 units.



b) The response to *u*(*t*) is . The response to *f*(*t*) is the sum of the responses to two step functions: .

**P21.1.5** Determine the LTs of the following functions: (a) (b) .



Solution: (a) L{} = == . Alternatively, ; L ; multiplying by  changes the transform to ; multiplying by  gives the same result.

(b) cos(4*t* – 1) = cos(4*t*)cos(1) + sin(4*t*)sin(1); hence, L {} = .

**P21.1.6** Determine the LT of .



**Solution:** L{} = . From the multiplication by *t* property, L{*t*} = .



**P21.2.2** Determine the LT of *f*(*t*) in Figure P21.2.2.

**Solution:** *f*(*t*) = 5*u*(*t* – 2) + 5*u*(*t* – 4) + 5*u*(*t* – 6) – 15*u*(*t* – 8);

F(s) = .



**P21.2.5** Determine the LT of: (a) *f*(*t*) in Figure P21.2.5; (b) *f*(1)(*t*).

**Solution:** (a) *Method 1:* *f*(*t*) =  ; hence,

.

*Method 2:*      .



(b) *L*{*df*/*dt*)} = *sF*(*s*) – *f*(0-) = .

As a check, *f*(1)(*t*) = *δ*(*t*) + 2*u*(*t*) – 3*u*(*t* – 1) – *δ*(*t* – 1) + *u*(*t* – 2). It follows that *L*{*df*/*dt*)} = .

Note that in order to obtain the Laplace transform of *f*(*t*) from *f*(1)(*t*), then because *f*(*t*) has an initial value of -1 at *t* = 0-, the correct expression is: *F*(*s*) =*L*{*df*/*dt*)} + . Thus, .

**P21.2.7** Determine the LT of the derivative of *f*(*t*) in Figure P21.2.7.



**Solution:** .





*L*{*df*/*dt*)} = sF(s) – f(0-) = .